From 2D Fractal Geometry to 3D Fractal Trigeometry

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It is a bit unclear what the generalization from a 2D Mandelbrot set (Fractal Geometry) to a 3D version (Fractal Trigeometry) should look like. So first, let's take a closer look at the two-dimensional case.

Fractal Geometry

Complex numbers are written as a + bi, where a and b are real numbers and i is the imaginary unit, with the property that $i^2 = -1$. This is enough to define elementary operations. What is important to us is the multiplication that looks like this:

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

Complex numbers can be represented as points in the so-called complex plane. The horizontal axis in this plane is called the real axis and the vertical axis is called the imaginary axis. For a complex number a + bi , a is called the real part and b is called the imaginary part. By taking the real part as a horizontal component and the imaginary part as a vertical component the complex number is fixed as a point in the complex plane. This is Cartesian notation.

If we have a complex number *a + bi* , the position in the complex plane can also be determined in another way, namely by means of polar notation:

$$a + bi = r (\cos \varphi + i \sin \varphi),$$

where r is the distance to the origin and φ the angle between the real axis and the line segment connecting the origin to the point in question (measured counterclockwise). The relationships between a, b, r and φ are given by:

$$r = \sqrt{a^2 + b^2},$$
$$a = r \cos \varphi,$$
$$b = r \sin \varphi.$$

In polar notation, multiplication is performed as follows:

$$r_1 (\cos \varphi_1 + i \sin \varphi_1)\, r_2 (\cos \varphi_2 + i \sin \varphi_2) = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$$

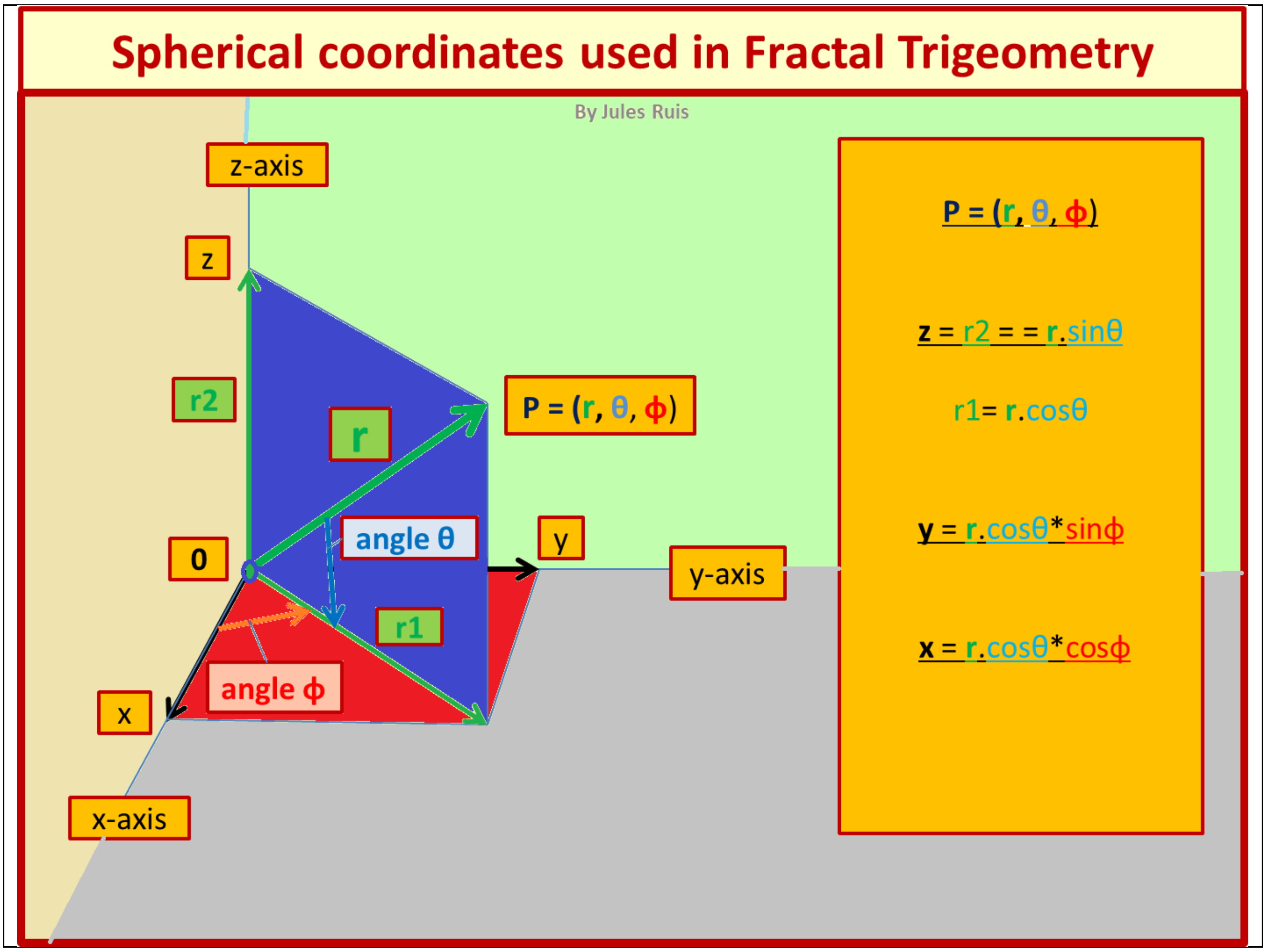
This follows simply from Euler's formula ($z = r.e^{i\varphi}$).

Now it is also clear what we need to do to square and generally take powers from a complex number. If $z = r (\cos \varphi + i \sin \varphi)$, then $z^n = r^n(\cos n\varphi + i \sin n\varphi)$. So to find the n^{th} power, the n^{th} power of the distance to the origin is taken as a new distance and the angle with the real axis is multiplied by ***n*** .

Fractal Trigeometry

Now let's look at the three-dimensional case. A point in three-dimensional space can be determined by choosing an x-, y- and z- coordinate , which is the Cartesian representation. Another possibility is to use spherical coordinates. In this case, a point (vector) is specified by the distance ***r*** to the origin (with r1 the projection of the vector on the xy-plane and r2 the projection of the vector on the z-axis), an angle **θ** as the angle in the z-r1 plane running from the vector to r1, and an angle ***φ*** as the angle in the xy-plane running from the x-axis to r1. The relationship between Cartesian coordinates and spherical coordinates is given by:

$r = \sqrt{x^2 + y^2 + z^2}$	$r1 = \sqrt{x^2 + y^2}$
$x = r (\cos \varphi \cos \theta)$	
$y = r (\sin \varphi \cos \theta),$	
$z = r (\sin \theta)$	



It is unclear how complex numbers in three dimensions should be defined, but with the geometric properties of complex numbers in two dimensions in mind, there is an obvious way to define a "multiplication". This then goes like this: to elevate to the n^{th} power, take the n^{th} power from the distance to the origin and multiply the angles φ and θ by n. This is only a generalization of the geometric way of multiplying complex numbers, and actually in three dimensions there is no longer a multiplication.

In this way we can try to construct a kind of 3D Mandelbrot set ; we call this the Juliusbulb set. It is very difficult to predict what the final Juliusbulb set will look like based on this construction alone. What is clear is that the z-axis plays a special role in this case. The x- and y-axis are actually similar to the axes in the complex plane and the angle φ plays the same role in the 3D case as in the 2D case. In 2D φ describes the rotation around the origin and in 3D φ describes the rotation around the z-axis. This could explain why the set in the z-direction is essentially different from the x- and y-direction.

Fractal Imaginator (FI) software

First created as BBM-15 d.d. 29 December 1997, see: www.fractal.org/BBM15.pdf

We want to calculate 3D fractals called the Juliusbulb (former Mandelbulb) and Juliabulb. Similar to the original 2D Mandelbrot the 3D formula is defined by $z \rightarrow z^n + c$ but where 'z' and 'c' are hypercomplex ('triplex') numbers representing Cartesian x-, y-, and z-coordinates.

The exponentiation term can be defined by:

$\{x,y,z\}^n = (r^n) \{ \cos(n*\phi) * \cos(n*\theta), \sin(n*\phi) * \cos(n*\theta), \sin(n*\theta)\}$

where $r = \mathrm{sqrt}(x^2 + y^2 + z^2)$ and $r1 = \mathrm{sqrt}(x^2 + y^2)$

As we define θ as the angle in z-r1-space and ϕ as the angle in x-y-space

then $\theta = \mathrm{atan2}(z / r1)$ so $\theta = \mathrm{atan2}(z / \mathrm{sqrt}(x^2 + y^2))$ and $\phi = \mathrm{atan2}(y/x)$

The addition term in $z \rightarrow z^n + c$ is similar to standard complex addition, and is simply defined by:

$(x,y,z\} + \{a,b,c) = \{x+a, y+b, z+c\}$

The rest of the algorithm is similar to the 2D Mandelbrot!

Summary Formula for 3D Juliusbulb and Juliabulb

$r = \mathrm{sqrt}(x^2 + y^2 + z^2)$

$\theta = \mathrm{atan2}(z / \mathrm{sqrt}(x^2 + y^2)$

$\phi = \mathrm{atan2}(y/x)$

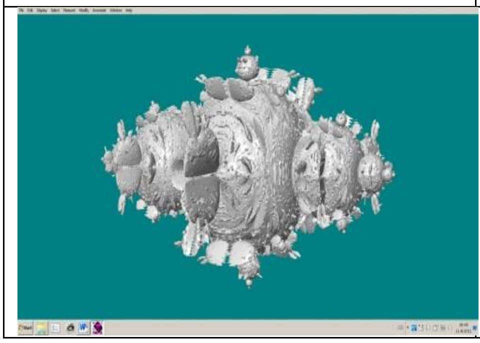
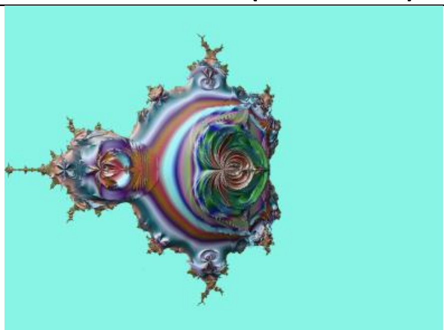
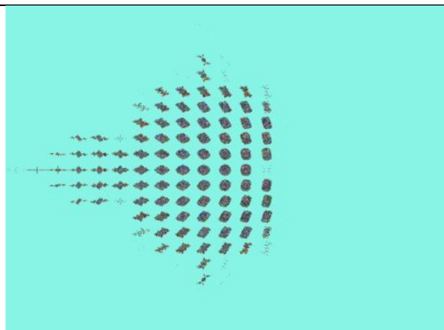
$newx = (r^n) * \cos(n*\phi) * \cos(n*\theta)$

$newy = (r^n) * \sin(n*\phi) * \cos(n*\theta)$

$newz = (r^n) * \sin(n*\theta)$

where n is the order of the 3D Juliusbulb (former Mandelbulb) and Juliabulb with formula $z \rightarrow z^n + c\#$.

Examples of 3D Juliabulb, 3D Juliusbulb (Mandelbulb) and 3D Julius Ruis Set

3D Juliabulb z^2	3D Juliusbulb z^2 (Mandelbulb)	3D Julius Ruis Set z^2
		

Fore more information see www.fractal.org

Box with 3D printed Fractal Juliusbulbs and Juliabulbs (printed at Shapeways).